Physics 566: Quantum Optics I

Problem Set 4
Due Thursday, September 26, 2013

Problem 1: Some Algebra with Density Matrices (10 Points)

(a) Prove some properties of the "trace" operation, $Tr(\hat{A}) \equiv \sum_{i=1}^{d} \langle e_i | \hat{A} | e_i \rangle$, where $\{|e_i\rangle\}$ is an orthonormal basis on a finite dimensional Hilbert space:

(i) $Tr(\hat{A})$ is independent of basis; $Tr\left(\hat{A}|\psi\rangle\langle\phi|\right) = \langle\phi|\hat{A}|\psi\rangle$; (ii) $Tr\left(\hat{A}\hat{B}\hat{C}\right) = Tr\left(\hat{C}\hat{A}\hat{B}\right)$.

(b) For a qubit whose density matrix is $\hat{\rho} = \frac{1}{2}\left(\hat{1} + \mathbf{Q}\cdot\hat{\vec{\sigma}}\right)$, where **Q** is the Bloch vector, show that $Tr\left(\hat{\rho}^2\right) = \frac{1}{2}\left(1 + |\mathbf{Q}|^2\right)$, and thus for a pure state |**Q**|=1, and the maximally mixed state |**Q**|=0.

(c) For the Mach-Zender interferometer Problem (2c) of P.S.#3, find the probability, $P_{1_{a,out}}$, to detect the photon in $|1_a;0_b\rangle$ at the output ports given the inputs in the state defined by each of the following density matrices:

$$(i)\ \hat{\rho}_{in} = |1_a,0_b\rangle\langle 1_a,0_b| ;\ (ii)\ \hat{\rho}_{in} = \frac{1}{2}|1_a,0_b\rangle\langle 1_a,0_b| + \frac{1}{2}|0_a,1_b\rangle\langle 0_a,1_b| ;$$

$$(iii)\ \hat{\rho}_{in} = \frac{1}{3}|1_a,0_b\rangle\langle 1_a,0_b| + \frac{2}{3}|0_a,1_b\rangle\langle 0_a,1_b| .$$

Sketch $P_{1_{a,out}}$ as a function of ϕ for each case and comment on your results.

Hint: You need to find $\hat{\rho}_{out}$ for each case, then determine $P_{1_{a,out}}$.

Problem 2: Ensemble Decomposition and the Density Matrix for Qubits (10 Points)

(a) Suppose we have a statistical mixture of spin 1/2 particles that consists of the state $\left|\uparrow_z\right\rangle$ with probability $\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)$ and the state $\left|\downarrow_z\right\rangle$ with probability $\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)$.

Find the matrix of the density operator in the basis $\{\left|\uparrow_z\right\rangle,\left|\downarrow_z\right\rangle\}$, and in the basis of eigenstates of $\hat{\sigma}_x$, $\{\left|\uparrow_x\right\rangle,\left|\downarrow_x\right\rangle\}$. What is the Bloch vector that describes this state?

(b) Now suppose we have a mixed state with 1/2 probability to have spin along $\mathbf{e}_{n_1}=\frac{1}{\sqrt{2}}\left(\mathbf{e}_z+\mathbf{e}_x\right)$ and 1/2 probability to have spin along $\mathbf{e}_{n_2}=\frac{1}{\sqrt{2}}\left(\mathbf{e}_z-\mathbf{e}_x\right)$. Is this a completely mixed state? Write the density operator in the basis $\{\left|\uparrow_z\right\rangle,\left|\downarrow_z\right\rangle\}$. Compare to part (a). Please comment on your result.

(c) Consider two ensembles:

- A statistical mixture of N pure states $\left\{\left|\uparrow_n\right\rangle\right\}$ with probabilities p_n,
- A statistical mixture of M pure states $\left\{\left|\uparrow_m\right\rangle\right\}$ with probabilities q_m.

Show that these mixture describe the *same* density operator $\hat{\rho}$ if

$$\mathbf{Q}=\sum_{n=1}^{N} p_n \mathbf{e}_n=\sum_{m=1}^{M} q_m \mathbf{e}_m ,$$

where $\mathbf{Q}$ is the Bloch vector for $\hat{\rho}$. Check this with your results of parts (a) and (b).

Problem 3: Inhomogeneous broadening (15 Points)

Many important phenomena occur in a time short compared to the "relaxation" time for coherent phenomena (e.g. spontaneous emission lifetime). Historically, these are known as coherent transients. Here are some examples.

(a) Free induction decay by inhomogeneous broadening: Consider a macroscopic ensemble of spins in a static magnetic field with a spatially inhomogeneous magnitude. The cloud is extended so that spins see an inhomogeneous distribution of B-fields in *z*-direction, with probability

$$P(B) = \frac{1}{\sqrt{2\pi(\delta B)^2}} e^{-\frac{(B-B_0)^2}{2(\delta B)^2}}$$

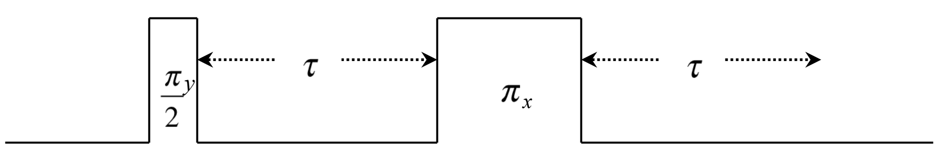
If the spins start in a coherent superposition of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, the sample will radiate magnetic dipole radiation at a mean frequency $\Omega_0 = \gamma B_0$ (γ being the gyromagnetic ratio), but the signal will decay much, much faster than the radiative decay rate. The decay arises because of the inhomogeneity -- different local spins will oscillate at different local frequencies. The resulting radiation from the different components eventually getting out of phase and destructively interfering.

(i) Calculate the characteristic decay time, known as T_2^*, due to inhomogeneity. Take $\gamma B_0 / 2\pi = 1$ MHz, $\gamma(\delta B)/2\pi = 10$ kHz

(ii) Calculate the spectrum of the radiation. The characteristic width $1/T_2^*$ is known as the inhomogeneous linewidth.

(iii) If the spins all start in $|\uparrow_z\rangle$, qualitatively describe how to achieve an approximate π/2-pulse to rotate all spins into the *x*-*y* plane (i.e. into a superposition state).

(b) Spin echo: Though inhomogeneous broadening will cause a decay of the ensemble averaged coherence, it is not a truly irreversible process. A procedure for recovering the coherence is known as a "spin echo". Consider the following pulse sequence.



The $\pi/2$-pulse about the *y*-axis acts according to (aiii) to bring all spins onto the *x*-axis of the Bloch sphere. For a time τ, the spins dephase. The π-pulse about the *x*-axis acts to time reverse the process. An "echo" signal will be seen at a time τ later.
Explain this process using this Bloch sphere. **Sketch** the signal one would detect of the radiated fields in rf coils.

(c) A spin-echo sequence is often used in a two-level Ramsey interferometer to make it more robust to inhomogeneities. Consider the following sequence

$\pi/2$ -*x*-rotation →Free evolution for time T/2 → π -*x*-rotation → Free evolution for time T/2 → $\pi/2$ -*x*-rotation

Explain the trajectory of the Bloch sphere. To what kind of inhomogeneities is this sequence this robust? This is a *direct* analogy to a Mach-Zender interferometer. Given what you said about the Ramsey interferometer, to what kinds of inhomogeneity is the Mach-Zender interferometer robust?